Calculation of ladder diagrams in arbitrary order

1983 J. Phys. A: Math. Gen. 16 2811

(http://iopscience.iop.org/0305-4470/16/12/026)

J. Phys. A: Math. Gen. 16 (1983) 2811–2816. Printed in Great Britain

Calculation of ladder diagrams in arbitrary order

V V Belokurov and N I Ussyukina
Institute of Nuclear Physics, Moscow State University, Moscow 117234, USSR

Received 17 January 1983

Abstract. A scalar ladder propagator-type diagram is rigorously calculated in an arbitrary order of perturbation theory.

Much attention has been given recently to calculating multiloop Feynman diagrams. It is stimulated by the efforts to make predictions of QCD more precise as well as by a deeper analysis of the structure of perturbation series in quantum field theory and attempts to sum it up.

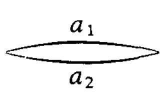
Several methods making possible the calculation of the renormalisation group (RG) quantities up to the three- or four-loop level have been developed (Celmaster and Gonsalves 1980, Chetyrkin *et al* 1980, Chetyrkin and Tkachov 1981, Curtright 1980, Vassiliev *et al* 1981). At the same time, attempts at further progress meet with serious difficulties. Here we present some simple arguments essentially enlarging the arsenal of methods for calculating Feynman diagrams.

Let us consider diagrams in φ^3-theory, as it is calculation of scalar integrals that is most difficult within this problem. We consider massless diagrams. There is a twofold reason for that. First, a power-like behaviour of the massless propagator both in momentum and coordinate spaces simplifies all the formulae greatly. Secondly, the massless propagator-type diagrams play a particular role in calculations of RG functions (see below).

Not to complicate the formulae, we omit henceforth the factors that are powers of 2, π, i. These factors can be easily restored in the final results.

We do the calculating in coordinate space. Each line of a diagram ($m = 0$) carries a power-like factor $1/(x^2 - \mathrm{i}0)^a$, that is pictured as $\underline{\quad a \quad}$.

The product and the convolution of lines in coordinate spaces are

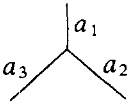


$$= \underline{\qquad a_1 + a_2 \qquad}, \qquad (1)$$

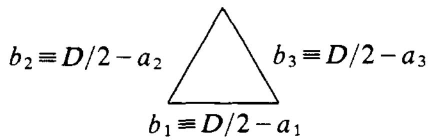
$$= \frac{\Gamma(D/2 - a_1)}{\Gamma(a_1)} \frac{\Gamma(D/2 - a_2)}{\Gamma(a_2)} \frac{\Gamma(a_1 + a_2 - D/2)}{\Gamma(D - a_1 - a_2)} \underline{\quad a_1 + a_2 - D/2 \quad},$$

D being the space–time dimension.

If $\Sigma a_i = D$, a three-line vertex is proportional to a triangle (D'Eramo *et al* 1971)



$$= \prod_{i=1}^{3} \frac{\Gamma(D/2-a_i)}{\Gamma(a_i)}$$

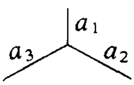


(2)

(Note that incidentally $\Sigma b_i = D/2$.) The three-line vertex and the triangle which have such lines are called unique.

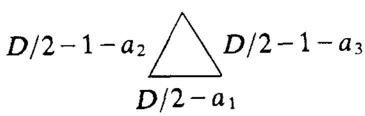
If a diagram contains a unique vertex or a unique triangle, its calculation is greatly simplified. Unfortunately, every line of a diagram is usually like $\underline{\quad 1+\alpha \quad}$, α being a parameter of some regularisation.

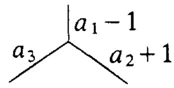
In that case the following identities are of use (Ussyukina 1983):

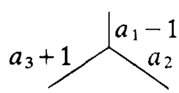
$\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)$ 

(if $\Sigma a_i = D-1$) (3)

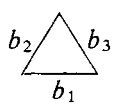
$= \Gamma(D/2-a_1)\Gamma(D/2-a_2-1)\Gamma(D/2-a_3-1)$



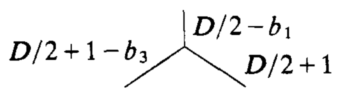
$-\Gamma(a_1-1)\Gamma(a_2+1)\Gamma(a_3)$ 

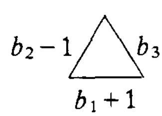
$-\Gamma(a_1-1)\Gamma(a_2)\Gamma(a_3+1)$ 

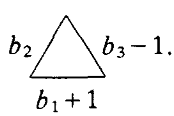
,

$\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)$ 

(if $\Sigma b_i = D/2+1$)

$= \Gamma(D/2-b_1)\Gamma(D/2-b_2+1)\Gamma(D/2-b_3+1)$ 

$-\Gamma(b_1+1)\Gamma(b_2-1)\Gamma(b_3)$ 

$-\Gamma(b_1+1)\Gamma(b_2)\Gamma(b_3-1)$ 

(4)

Let us apply these identities to calculate multiloop diagrams. Propagator-type diagrams are of special interest because of the following remarkable fact: the problem of calculating the counterterm of an arbitrary L-loop diagram with arbitrary masses and an arbitrary number of external momenta within the MS (minimal subtraction see 't Hooft (1973)) scheme can be reduced to the problem of calculating to $O(\varepsilon^0)$ some $(L-1)$-loop massless integrals with only one external momentum (Vladimirov 1980, Chetyrkin *et al* 1980).

Let us consider a scalar N-loop ladder diagram of propagator type, pictured in figure 1(a) ($D=4$). This is a convergent diagram. Denote it as Φ_N. To calculate Φ_N, it is convenient to use an auxiliary regularisation: $\Phi_N(\alpha_1, \alpha_2, \alpha_3)$ (see figure 1(b)), $\Phi_N = \Phi_N(0, 0, 0)$. Let the parameters obey the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{5}$$

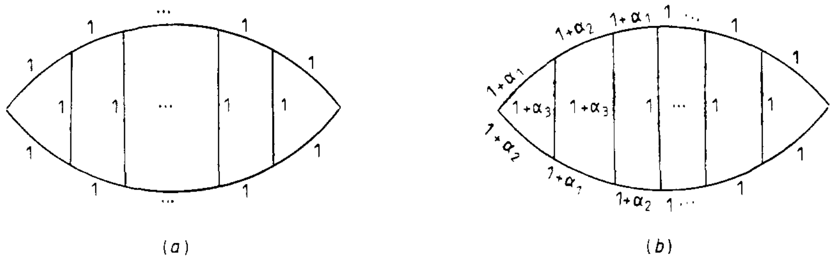


Figure 1. (a) Scalar N-loop propagator-type ladder diagram (Φ_N). (b) Special analytic regularisation of diagram of (a) ($\Phi_N(\alpha_1, \alpha_2, \alpha_3)$).

A simple recurrence relation relating $\Phi_N(\alpha_1, \alpha_2, \alpha_3)$ to $\Phi_{N-1}(\alpha_1, \alpha_2, \alpha_3)$ is proved to exist:

$$\Phi_N(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\alpha_1\alpha_2\alpha_3}\left(\frac{1+\alpha_2}{}\,\alpha_1 \mathscr{T}\Phi_{N-1}(\alpha_1, 0, -\alpha_1) + \frac{1+\alpha_1}{}\,\alpha_2\mathscr{T}\Phi_{N-1}(0, \alpha_2, -\alpha_2) + \frac{1}{}\,\alpha_3\Phi_{N-1}(\alpha_1, \alpha_2, \alpha_3)\right), \tag{6}$$

$$\mathscr{T} \equiv \prod_{i=1}^{3} \Gamma(1-\alpha_i)/\Gamma(1+\alpha_i) \qquad N \geqslant 4.$$

To prove this equation, consider a graph shown in figure 2. With the help of (1)–(5) this graph can be reduced to the linear combination of graphs (a)–(c) of figure 3 with coefficients $\mathscr{T}/\alpha_1\alpha_3$, $\mathscr{T}/\alpha_2\alpha_3$, $1/\alpha_1\alpha_2$. Equation (6) follows readily from that reduction.

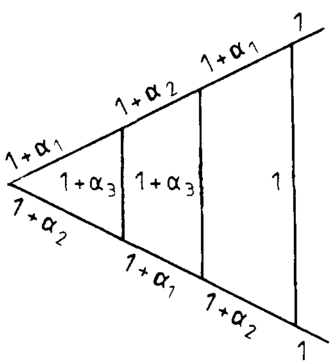


Figure 2. Ladder vertex graph that is used to prove (6).

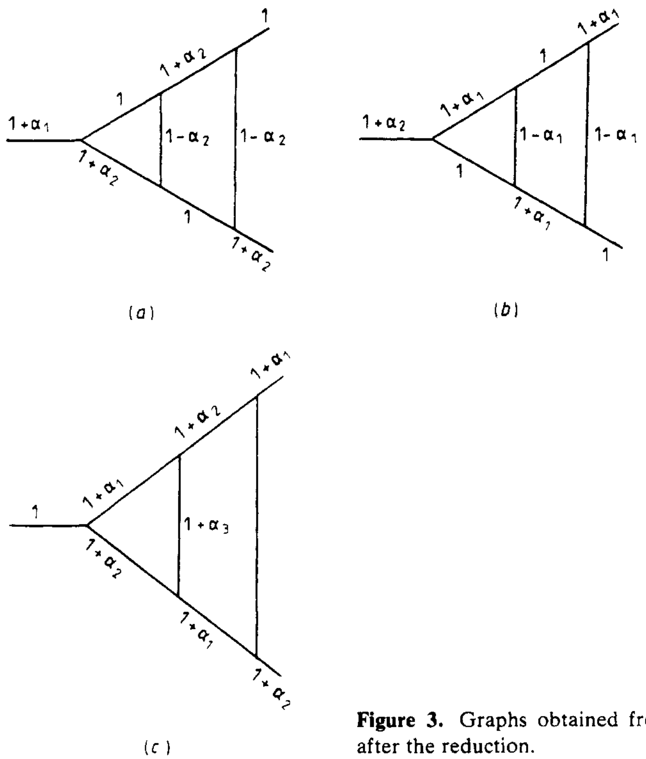


Figure 3. Graphs obtained from that of figure 2 after the reduction.

To calculate $\Phi_2(\alpha_1, \alpha_2, \alpha_3)$ consider the diagram shown in figure 4. Let $D = 4 - 2\varepsilon$ for the time being. This diagram can be calculated using (2)–(4) and in the limit $\varepsilon \to 0$ yields

$$\Phi_2(\alpha_1, \alpha_2, \alpha_3) = -\frac{1-\alpha_3}{} \frac{\mathscr{T}}{\alpha_1\alpha_2\alpha_3}\{\alpha_1[\Psi(1+\alpha_1)+\Psi(1-\alpha_1)] + \alpha_2[\Psi(1+\alpha_2)+\Psi(1-\alpha_2)]+\alpha_3[\Psi(1+\alpha_3)+\Psi(1-\alpha_3)]\}. \tag{7}$$

As the three α obey equation (5), it is convenient to rewrite (7) in the form

$$\Phi_2(\alpha_1, \alpha_2, \alpha_3) = -\frac{1-\alpha_3}{} \frac{\mathscr{T}}{\alpha_1\alpha_2\alpha_3}\sum_{n=0}^{\infty} a_n(\alpha_1^{2n+3}+\alpha_2^{2n+3}+\alpha_3^{2n+3})$$

$$= -\frac{1}{(p^2)^{1+\alpha_3}}\frac{\Gamma(1+\alpha_3)}{\Gamma(1-\alpha_3)}\mathscr{T}\sum_{n=0}^{\infty} a_n \sum_{k=1}^{2n+2} c_{2n+3}^{k}\frac{\alpha_1^{k-1}\alpha_2^{2n+2-k}}{\alpha_1+\alpha_2}, \tag{8}$$

$$a_n \equiv 2\Psi^{(2n+2)}(1)/(2n+2)! = -2\zeta(2n+3), \qquad c_n^k \equiv \frac{n!}{k!(n-k)!}.$$

An expression for $\Phi_3(\alpha_1, \alpha_2, \alpha_3)$ can be obtained in a way similar to that of equation (6). The result is

$$\Phi_3(\alpha_1, \alpha_2, \alpha_3) = \frac{1-\alpha_3}{} \frac{\mathscr{T}}{\alpha_1\alpha_2\alpha_3}\Big\{\alpha_1\Phi_2(\alpha_1, -\alpha_1, 0)+\alpha_2\Phi_2(-\alpha_2, \alpha_2, 0) + \alpha_3\mathscr{T}\Big[\Phi_2(-\alpha_3, -\alpha_2, -\alpha_1)\otimes\frac{-\alpha_1}{}\Big]\Big\}. \tag{9}$$

The last term corresponds to the diagram shown in figure 5.

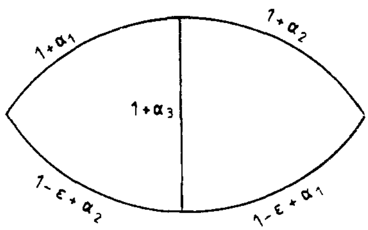


Figure 4. Two-loop propagator-type diagram $(\Phi_2(\alpha_1, \alpha_2, \alpha_3))$.

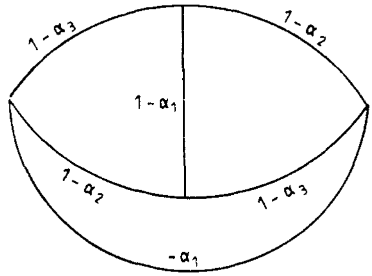


Figure 5. Graph corresponding to the last term in (9).

On account of (8), equation (9) leads to

$$\Phi_3(\alpha_1, \alpha_2, \alpha_3) = -\frac{1}{(p^2)^{2+\alpha_3}} \frac{\Gamma(1+\alpha_3)}{\Gamma(1-\alpha_3)} \mathcal{T}^2 \sum_{n=1}^{\infty} a_n \sum_{k=2}^{2n+1} c_{2n+3}^{k} \frac{\alpha_1^{k-2} \alpha_2^{2n+1-k}}{\alpha_1+\alpha_2}. \tag{10}$$

Now from (6) we can obtain an expression for $\Phi_N(\alpha_1, \alpha_2, \alpha_3)$ for all N. It has the form

$$\Phi_N(\alpha_1, \alpha_2, \alpha_3) = -\frac{1}{(p^2)^{N-1+\alpha_3}} \frac{\Gamma(1+\alpha_3)}{\Gamma(1-\alpha_3)} \mathcal{T}^2 \times \sum_{n=N-2}^{\infty} a_n \sum_{k=N-1}^{2n+3-(N-1)} c_{2n+3}^{k} \frac{\alpha_1^{k-N+1} \alpha_2^{2n+3-k-(N-1)}}{\alpha_1+\alpha_2}. \tag{11}$$

The proof of (11) can be done easily by induction. When all α tend to zero, only the term with $n = N-2$ is essential in this sum:

$$\Phi_N = -\frac{1}{(p^2)^{N-1}} a_{N-2} \lim_{\alpha_1, \alpha_2 \to 0} \left(c_{2N-1}^{N-1} \frac{\alpha_2}{\alpha_1+\alpha_2} + c_{2N-1}^{N} \frac{\alpha_1}{\alpha_1+\alpha_2} \right).$$

Thus

$$\Phi_N = \frac{1}{(p^2)^{N-1}} 2\zeta(2N-1) c_{2N-1}^{N}. \tag{12}$$

The above calculation seems to be the first example of calculations of non-trivial diagrams in an arbitrary order. We stress that no additional assumption is used here.

The method described can be used to calculate divergent diagrams as well. The divergences will manifest themselves as poles in the regularisation parameters (Speer 1968).

References

Celmaster W and Gonsalves R 1980 *Phys. Rev.* D **21** 3112
Chetyrkin K G, Kataev A L and Tkachov F V 1980 *Nucl. Phys.* B **174** 345
Chetyrkin K G and Tkachov F V 1981 *Nucl. Phys.* B **192** 159
Curtright T 1980 *Phys. Rev.* D **21** 1543

D'Eramo M, Peliti L and Parisi G 1971 *Lett. Nuovo Cimento* **2** No 26, 878
't Hooft G 1973 *Nucl. Phys.* B **61** 455
Speer E 1968 *J. Math. Phys.* **9** 1404
Ussyukina N I 1983 *Teor. Mat. Fiz.* **54** No 1, 124 (in Russian)
Vassiliev A N, Pis'mak Yu M and Honkonen Yu R 1981 *Teor. Mat. Fiz.* **47** 291 (in Russian)
Vladimirov A A 1980 *Teor. Mat. Fiz.* **43** 210 (in Russian)